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Monodromy Matrix for Linear Difference Operators with Almost Constant Coefficients

AURELIJA TRGO*

Department of Mathematics, Carnegie Mellon University, Pittsburgh, Pennsylvania 15213

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A new method for solving the discrete scattering problem for a linear single-valued difference operator of arbitrary order with almost constant coefficients is proposed. The treatment is concerned with the asymptotic behavior of its eigenfunctions as $|t| \to \infty$. The purpose of the paper is to investigate the transition between the asymptotically free states of the underlying system, defined in terms of the monodromy operator. A rare mathematical tool is used: *an infinite matrix product.* It is shown that, if the coefficients of the aforementioned operator are suitably behaved, the monodromy operator exists in the form of the convergent two-sided infinite product of matrices associated with the matrix eigenvalue equation corresponding to the scattering problem. The application of the method to the solution of the scattering problem for Toda lattice equations is demonstrated. However, the approach is quite general and should be applicable to other forms of lattice equations.

1. INTRODUCTION

It is well known that certain discrete integrable Hamiltonian systems arising in mechanics and physics and governed by discrete nonlinear evolution equations are solvable by means of scattering and inverse scattering theory. One of the most significant examples is undoubtedly the *Toda lattice.*

Motivated by these results, in this work we propose a new method for solving the direct scattering problem associated with a linear difference operator of arbitrary order with *almost constant coefficients.* The notion of a *monodromy operator* (monodromy matrix) is introduced to describe the transition between the asymptotics of the eigenfunctions as $t \to \infty$ and

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$t \to -\infty$. We show that, under certain assumptions, this matrix can be represented in the form of a (two-sided) *infinite product of matrices* associated with the aforementioned difference operator. The notion of such a product is briefly discussed in Section 2. Our approach via a first-order system greatly simplifies the analysis and enables us to establish this particular property of the monodromy operator. As an additional result, we present a procedure for analysing the behavior of the monodromy operator under the influence of shift.

Finally, we rely on the work of Flaschka [9] and Toda [13], in order to demonstrate the application of our results to the second-order case. Namely, we review the discrete Schrödinger scattering problem with specific application to the Toda lattice. We show in particular how to recover the *scattering matrix*, that is usually considered in association with this problem, from the monodromy matrix.

For general n the essential analysis of our scattering problem will actually involve only the bounded eigenfunctions. However, it turns out that the second-order selfadjoint difference operator associated with the Toda lattice equations possesses the l^2-eigenfunctions. A proof of this fact follows along the lines of a similar result in continuous theory (see [3, 9]).

2. Preliminaries

Let $H = l^\infty(\mathbf{Z})$ be the Banach space of bounded, doubly infinite complex sequences $\{y(t); -\infty < t < \infty\}$. In the sequel we denote by $\mathbf{E}$ and $\mathbf{E}^{-1}$ the *forward* and *backward shift* (or translation) operators acting on H; that is, we have $\mathbf{E}^k\, y(t) = y(t + k)$ for all integers k.

We consider the nth order linear difference operator $\mathbf{L}$ defined by

$$\mathbf{L} = \tilde{p}_{-k}(t)\mathbf{E}^{-k} + \tilde{p}_{-k+1}(t)\mathbf{E}^{-k+1} + \cdots + \tilde{p}_{m-1}(t)\mathbf{E}^{m-1} + \tilde{p}_m(t)\mathbf{E}^m, \qquad (2.1)$$

where $n = k + m$, $n \geq 2$, provided that $\tilde{p}_{-k}(t)\tilde{p}_m(t) \neq 0$. By hypothesis,

$$\tilde{p}_j(t) = p_j + p_j(t),$$

where the family of functions $\{p_j(t)\}$ will be regarded as the (generalized) potential $V(t)$. *We make in particular the standing assumption that, unless otherwise stated, $V(t)$ is assumed to decay sufficiently rapidly as $|t| \to \infty$.* However, we shall see below that such a restrictive assumption can be relaxed to a certain extent and still be sufficient for most of our conclusions to remain true.

Our treatment of the direct scattering problem for the operator $\mathbf{L}$ centers on the following: *given* $\lambda \in \sigma(\mathbf{L})$, *we study the asymptotic behavior of the solutions to the eigenvalue equation*

$$\mathbf{L}y(t) = \lambda y(t) \tag{2.2}$$

in the range $-\infty < t < \infty$, *in order to identify the corresponding asymptotically free states as* $t \to +\infty$ *and* $t \to -\infty$. The transition between these states will be described in terms of the monodromy operator (monodromy matrix). We show that, if the potential $V(t)$ decays as fast as $(|t|)^{-1-\varepsilon}$ for some $\varepsilon > 0$, then this operator exists, is invertible, and can be represented in the form of an infinite product of matrices associated with Eq. (2.2).

Remark 1. The point spectrum of the Hamiltonian corresponds to the energy levels of bound states of the system. These states give rise to special *soliton* solutions. The solitons (solitary waves) move around the system at different speeds and due to special collision properties regain their shapes and speeds after interactions. That is, the solitons will never collapse. The complete analytical understanding of the existence of the soliton phenomena for general discrete systems governed by the difference equations of the order higher than two lies beyond the scope of this work and remains an interesting open problem. Up until now, apart from some numerical studies of solitons (see, e.g., [7, 8]), the only examples of discrete systems known to possess solitary waves have been carefully constructed integrable Hamiltonian systems.

Before beginning our systematic study of the problem (2.2), we first introduce the notion of an infinite product of matrices and briefly discuss its convergence conditions.

The notion of an infinite product of matrices belonging to $\mathbf{C}^{n\times n}$ is analogous to that for numbers. However, because of the multiplication structure in Banach algebra $\mathbf{C}^{n\times n}$, a somewhat careful approach is needed when dealing with this infinite product.

Let $\{\mathbf{A}_k\}_{k\in\mathbf{Z}}$ be any sequence in $\mathbf{C}^{n\times n}$. We shall consider here the infinite matrix products in the form

$$\prod_{k=-\infty}^{\infty} (\mathbf{I} + \mathbf{A}_k) \tag{2.3}$$

and

$$\prod_{k=-\infty}^{-1} (\mathbf{I} + \mathbf{A}_k), \qquad \prod_{k=0}^{\infty} (\mathbf{I} + \mathbf{A}_k), \tag{2.4}$$

under the assumption that $\det(\mathbf{I} + \mathbf{A}_k) \neq 0$ for all $k \in \mathbf{Z}$. We shall continue to refer to the infinite product given by (2.3) as a *two-sided* infinite product of matrices, whereas both products in (2.4) will be called *one-sided* products.

Technical Remark. Unless we specify otherwise, the ordering of factors in all our matrix products (finite or infinite) will always be from the right to the left; that is,

$$\prod_{k=i}^{j} \mathbf{B}_k = \mathbf{B}_j \mathbf{B}_{j-1} \cdots \mathbf{B}_i, \qquad i < j.$$

In our further discussion we use the *spectral* norm in $\mathbf{C}^{n \times n}$ (denoted by $\|\cdot\|$). The following properties of this norm will be needed later.

LEMMA 2.1. *If* $\mathbf{U}$ *is a unitary matrix then* $\|\mathbf{U}\| = 1$. *Moreover, for any matrix* $\mathbf{A}$, $\|\mathbf{A}\|$ *is invariant under unitary similarity transformations.*

Associated with the infinite product (2.3), there is a sequence of (two-sided) *partial products* $q^{(n,m)}$ given as

$$q^{(n,m)} = \prod_{k=n}^{m} (\mathbf{I} + \mathbf{A}_k), \qquad n < m.$$

Accordingly, in the case of the infinite products in (2.4) we consider the (one-sided) partial products $q^{(n,-1)}$ $(n < -1)$ and $q^{(0,m)}$ $(m > 0)$, respectively.

DEFINITION 2.1. The two-sided infinite product $\prod_{k=-\infty}^{\infty} (\mathbf{I} + \mathbf{A}_k)$ is said to *converge* to the matrix

$$\mathbf{A} = \lim_{(n,m) \to (-\infty, \infty)} q^{(n,m)}$$

if this limit exists and is nonsingular. In this case we write

$$\mathbf{A} = \prod_{k=-\infty}^{\infty} (\mathbf{I} + \mathbf{A}_k).$$

The convergence of a one-sided infinite matrix product is defined in the same manner.

We note that the convergence of the two-sided product (2.3) implies the convergence of both products in (2.4), and vice versa. Accordingly, in order to derive the convergence conditions for all of these products, it suffices to consider the product

$$\prod_{k=0}^{\infty} (\mathbf{I} + \mathbf{A}_k). \tag{2.5}$$

A necessary condition for its convergence is $\mathbf{A}_k \to 0$ as $k \to \infty$.

We further say that the infinite product (2.5) converges *absolutely* if the corresponding (number) product

$$\prod_{k=0}^{\infty} (1 + \|\mathbf{A}_k\|) \tag{2.6}$$

converges (to a *nonzero* value). It is readily verified that, on the assumption $\det(\mathbf{I} + \mathbf{A}_k) \neq 0$ for all $k \geq 0$, the absolute convergence of the product (2.5) implies its convergence.

THEOREM 2.1. *Let* $\det(\mathbf{I} + \mathbf{A}_k) \neq 0$ *for all* $k \geq 0$. *The convergence of the (number) series* $\sum_{k=0}^{\infty} \|\mathbf{A}_k\|$ *is a sufficient condition for the absolute convergence of the infinite product* $\prod_{k=0}^{\infty} (\mathbf{I} + \mathbf{A}_k)$.

3. Matrix Formalism

We return to a detailed discussion of the eigenvalue problem (2.2). To simplify the presentation, we first turn our attention to the case of the one-sided operators.

Given a two-sided operator $\mathbf{L}$ in (2.1), we associate with $\mathbf{L}$ the translated operator $\mathbf{L}^{(k)}$ defined by

$$\mathbf{L}^{(k)} = \mathbf{L}\mathbf{E}^k = \sum_{j=0}^{n} \tilde{q}_j(t)\mathbf{E}^j,$$

where we set $\tilde{q}_j \equiv \tilde{p}_{j-k}$ for all $0 \leq j \leq n$. On the assumption $\tilde{q}_n \equiv 1$, the corresponding eigenvalue equation $\mathbf{L}^{(k)}y = \lambda y$ takes therefore an expanded form as

$$y(t+n) + q_{n-1}y(t+n-1) + \cdots + q_1 y(t+1) + (q_0 - \lambda)y(t) = r(t), \tag{3.1}$$

where $r(t) = -\sum_{j=0}^{n-1} q_j(t)y(t+j)$. This nth order scalar difference equation can be reduced to a linear system of the form

$$\mathbf{x}(t+1, \lambda) = [\mathcal{A}_\lambda + \mathcal{B}(t)]\mathbf{x}(t, \lambda), \tag{3.2}$$

where

$$\mathcal{A}_\lambda = \begin{bmatrix} 0 & 1 & 0 & & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & \ddots & & \vdots \\ \vdots & & & & \ddots & 0 \\ 0 & \cdots & \cdots & & & 1 \\ \lambda - q_0 & -q_1 & \cdots & \cdots & & -q_{n-1} \end{bmatrix} \tag{3.3}$$

and the "potential" matrix $\mathcal{B}(t)$ is given by

$$\mathcal{B}(t) = e_n \mathbf{q}^T(t), \tag{3.4}$$

where $\mathbf{q}(t) = [-q_0(t), -q_1(t), ..., -q_{n-1}(t)]^T$. Note that $\mathcal{B}(t)$ is λ-independent.

In order to solve the direct scattering problem associated with the matrix eigenvalue equation (3.2), we seek those λ for which bounded vector solutions to Eq. (3.2) exist for all t, subject to some prescribed initial condition $\mathbf{x}(t_0, \lambda) = \text{const} \neq 0$. This, in turn, implies that a certain set of restrictions must be placed on matrices $\mathcal{A}_\lambda$ and $\mathcal{B}(\mathbf{t})$. Accordingly, we suppose henceforth in our discussion that the vector equation (3.2) satisfies the following assumptions:

A-I. $[\mathcal{A}_\lambda + \mathcal{B}(t)]$ is nonsingular for all t;

A-II. $\mathcal{A}_\lambda$ has distinct eigenvalues, say $z_1(\lambda), z_2(\lambda), ..., z_n(\lambda)$;

A-III. $|z_j(\lambda)| = 1, 1 \leq j \leq n$.

PROPOSITION 3.1. *For* (A-III) *to hold, it is necessary that* $|\lambda - q_0| = 1$ *and* $|q_j| \leq \binom{n}{j}$, $1 \leq j \leq n - 1$.

Proof. From the general properties of the Frobenius matrices we derive

$$\det(z\mathbf{I} - \mathcal{A}_\lambda) = z^n + q_{n-1}z^{n-1} + \cdots + q_1 z + q_0 - \lambda.$$

On the other hand, since $\det(z\mathbf{I} - \mathcal{A}_\lambda) = \prod_{i=1}^n (z - z_i(\lambda))$, one finds that, for $0 < j \leq n$,

$$(-1)^{n-j} q_j = \sum_{1 \leq i_1 < i_2 < \cdots < i_{n-j} \leq n} z_{i_1}(\lambda) z_{i_2}(\lambda) \cdots z_{i_{n-j}}(\lambda)$$

and

$$q_0 - \lambda = (-1)^n \prod_{i=1}^{n} z_i(\lambda).$$

The assertion now follows easily from (A-III). ∎

Remark 2. Since the zeros of a polynomial depend continuously on its coefficients, it is intuitively plausible that the assumption A-II holds for all but finitely many values of λ.

Hereafter in this work, the eigenvalue λ is supposed to be fixed. In view of the assumption (A-II), the matrix $\mathscr{A}_\lambda$ is *diagonalizable.* In fact,

$$\mathbf{V}_\lambda^{-1}\mathscr{A}_\lambda\mathbf{V}_\lambda = \mathscr{I}_\lambda, \tag{3.5}$$

where the matrix $\mathbf{V}_\lambda$ is the Vandermonde matrix $\mathbf{V}_\lambda(z_1(\lambda), z_2(\lambda), \ldots, z_n(\lambda))$ and $\mathscr{I}_\lambda = \operatorname{diag}(z_1(\lambda), z_2(\lambda), \ldots, z_n(\lambda))$. Clearly, such a $\mathscr{I}_\lambda$ is *unitary.* It will be convenient to transform our vector equation (3.2) to take advantage of the diagonalization (3.5). If we change variable

$$\mathbf{u}(t, \lambda) = \mathbf{V}_\lambda^{-1}\mathbf{x}(t, \lambda) \tag{3.6}$$

and set

$$\mathbf{D}(t) = \mathbf{V}_\lambda^{-1}\mathscr{B}(t)\mathbf{V}_\lambda, \tag{3.7}$$

Eq. (3.2) becomes

$$\mathbf{u}(t + 1, \lambda) = [\mathscr{I}_\lambda + \mathbf{D}(t)]\mathbf{u}(t, \lambda). \tag{3.8}$$

As is obvious from (A-I), $\det[\mathscr{I}_\lambda + \mathbf{D}(t)] \neq 0$ for all t. Hence, if we prescribe the initial condition $\mathbf{u}(0, \lambda) = \mathbf{u}_0$ then any solution of Eq. (3.8) is uniquely determined by

$$\mathbf{u}(t, \lambda) = \begin{cases} [\prod_{i=0}^{t-1} (\mathscr{I}_\lambda + \mathbf{D}(i))]\mathbf{u}_0 & \text{if } t > 0 \\ [\prod_{i=t}^{-1} (\mathscr{I}_\lambda + \mathbf{D}(i))^{-1}]\mathbf{u}_0 & \text{if } t < 0. \end{cases} \tag{3.9}$$

Note that in the case $t < 0$ we obtain a development of the form

$$\mathbf{u}(t, \lambda) = (\mathcal{J}_\lambda + \mathbf{D}(t))^{-1}(\mathcal{J}_\lambda + \mathbf{D}(t+1))^{-1} \cdots (\mathcal{J}_\lambda + \mathbf{D}(-1))^{-1}\mathbf{u}_0, \qquad t < 0.$$

That is, the ordering of factors is from the left to the right.

Let us denote by $\mathcal{H}$ the (linear) space of solutions to the matrix eigenvalue equation (3.8). We wish to establish a sufficient condition for all $\mathbf{u} \in \mathcal{H}$, and in particular for the solution $\mathbf{x}$ itself, to remain bounded for all t. Note that thus far in our analysis we have imposed no "constitutive assumptions" on the matrix $\mathcal{B}(t)$ (other than the assumption made at the very beginning that the potential $V(t) = \{q_j(t), 0 \leq j \leq n-1\}$ decays sufficiently rapidly as $|t| \to \infty$). Let us henceforth suppose $V(t)$ satisfies the decay condition

$$q_j(t) = O(|t|^{-1-\varepsilon}), \qquad 0 \leq j \leq n-1 \tag{3.10}$$

as $|t| \to \infty$ for some $\varepsilon > 0$. Such V will be called a *short-range potential.*

THEOREM 3.1. *Consider* $\mathbf{x}(t, \lambda)$ *satisfying the equation* $\mathbf{x}(t+1, \lambda) = [\mathcal{A}_\lambda + \mathcal{B}(t)]\mathbf{x}(t, \lambda)$ *with the assumptions* (A-I)–(A-III) *applied. If* $V(t)$ *is a short-range potential then the solution* $\mathbf{x}(t, \lambda)$ *is bounded for all* t.

To prove this theorem the following lemma, known as Hukurawa's theorem, is useful; a proof can be found in Miller [12].

LEMMA 3.1. *Let* $\boldsymbol{\eta}(t)$ *satisfy the vector equation*

$$\boldsymbol{\eta}(t+1) = [\mathbf{A}_1 + \mathbf{A}_2(t)]\boldsymbol{\eta}(t)$$

for all t. *Suppose that all solutions of* $\boldsymbol{\eta}(t+1) = \mathbf{A}_1\boldsymbol{\eta}(t)$ *are bounded and*

$$\sum_{t=-\infty}^{\infty} \|\mathbf{A}_2(t)\| < \infty.$$

Then the solution $\boldsymbol{\eta}(t)$ *is bounded for all* t.

Proof of Theorem 3.1. By virtue of the change-of-variable formula (3.6), it suffices to show that, under the given assumptions, Eq. (3.8) has bounded solutions. It follows from (A-II) and (A-III) that any solution of the equation $\mathbf{u}(t+1, \lambda) = \mathcal{J}_\lambda\mathbf{u}(t, \lambda)$ is bounded for all t. If the estimate (3.10) holds then, for every $0 \leq j \leq n-1$, the function $q_j(t)$ satisfies

$$\sum_{t=-\infty}^{\infty} |q_j(t)| \leq C \sum_{t=0}^{\infty} \frac{1}{|t|^{1+\varepsilon}} < \infty$$

for some constant C. Thus, as is clear from Eqs. (3.4) and (3.7),

$$\sum_{t=-\infty}^{\infty} \|\mathbf{D}(t)\| < \infty, \tag{3.11}$$

and therefore Lemma 3.1 completes the proof. ∎

COROLLARY 3.1. *In view of the assumptions* (A-1)–(A-III), *every* $y(t, \lambda)$ *satisfying Eq.* (3.1) *remains bounded as* $|t| \to \infty$, *provided that* (3.10) *holds.*

4. ASYMPTOTICS OF EIGENFUNCTIONS; MONODROMY MATRIX

In the dynamic theory of scattering the behavior of the solution of the perturbed system for large $|t|$ is compared with the corresponding behavior of solutions of the unperturbed system.

The starting point for our approach to scattering problem for the operator $\mathbf{L}$ is to analyse the asymptotic properties of $\mathbf{u} \in \mathcal{H}$. This choice is motivated by the great simplification in our formulas and the transparency of the results that is evident from the change-of-variable formula (3.6). For the sake of notation, in everything that follows the explicit dependence on the (fixed) variable λ will be omitted and only used when needed to emphasize the eigenvalue dependence.

The vector equation (3.8) provides asymptotic information about $\mathbf{u}(t)$ in the following sense: when $V(t) \to 0$ sufficiently rapidly as $|t| \to \infty$, Eq. (3.8) behaves asymptotically like the *free* (unperturbed) equation

$$\mathbf{u}^{(0)}(t+1) = \mathcal{J}\mathbf{u}^{(0)}(t), \tag{4.1}$$

which corresponds to $\mathbf{D}(t) \equiv 0$. This equation has a basis of solutions

$$\mathbf{u}_k^{(0)}(t) = z_k^t \mathbf{e}_k, \qquad 1 \leq k \leq n, \tag{4.2}$$

where $\mathbf{e}_k$ are the unit vectors in $\mathbf{C}^n$ ($z_k = z_k(\lambda)$).

We introduce now a (scalar) solution of the eigenvalue equation, Eq. (3.1), called the *Jost solution* and denoted by $\varphi_k(\cdot, \lambda)$, by demanding that

$$\lim_{t \to \pm\infty} z_k^{-t} \varphi_k(t) = 1 \tag{4.3}$$

for every $1 \leq k \leq n$, so that the function $\mathbf{u}(t) = [\varphi_1(t), \varphi_2(t), \ldots, \varphi_n(t)]^T$ reduces to a solution of the free equation (4.1) as $|t| \to \infty$. According to this, it follows that the Jost solutions φ_k, $1 \leq k \leq n$, constitute a set of n linearly independent solutions of (3.1), since their Casoratian, by virtue of

(A-II), is nonzero. We shall see later that, under the condition (3.10), such solutions exist and moreover, they form a distinguished basis of solutions to our scattering problem. More specifically, it will be shown that for large $|t|$, every solution of the perturbed equation (3.1) can be written as a linear combination, with constant (t-independent) coefficients, of these Jost solutions.

Let $\mathbf{e}_+ = \mathbf{e}_+(\lambda) \in \mathbf{C}^n$. We say that $\mathbf{e}_+$ is an *asymptotically free state* for the operator $\mathcal{J} + \mathbf{D}(t)$ at $+\infty$, if there is a $\mathbf{g} \in \mathbf{C}^n$, $\mathbf{g} = \mathbf{g}(\lambda)$, such that

$$\prod_{i=0}^{t-1} (\mathcal{J} + \mathbf{D}(i))\mathbf{g} = \mathcal{J}^t\mathbf{e}_+ + o(1), \qquad t \to \infty. \tag{4.4}$$

Similarly, we consider vector $\mathbf{e}_- = \mathbf{e}_-(\lambda)$ to be an asymptotically free state of this operator at $-\infty$ if

$$\left[\prod_{i=t}^{-1} (\mathcal{J} + \mathbf{D}(i))\right]^{-1} \mathbf{g} = \mathcal{J}^t\mathbf{e}^- + o(1), \qquad t < 0 \tag{4.5}$$

as $t \to -\infty$. Thus, to each asymptotically free state $\mathbf{e}_\pm$ there corresponds an initial condition $\mathbf{g}$ of the perturbed equation (3.8) (i.e., $\mathbf{u}(t + 1) = [\mathcal{J} + \mathbf{D}(t)]\mathbf{u}(t)$) such that the solution $\mathbf{u}(t)$ behaves like a respective solution $\mathbf{u}^{(0)}(t)$ of the free equation (4.1) as $t \to \infty$ and $t \to -\infty$ which satisfies $\mathbf{u}^{(0)}(0) = \mathbf{e}_\pm$.

Let $\mathcal{X}_+$ and $\mathcal{X}_-$ denote the asymptotically free states of the operator $\mathcal{J} + \mathbf{D}(t)$ as $t \to \infty$ and $t \to -\infty$, respectively. The problem (3.1) and (4.3) becomes, for our system (3.8): given $\mathbf{u} \in \mathcal{H}$, determine necessary and sufficient conditions so that

$$\lim_{t \to \pm\infty} \mathbf{J}^{-t}\mathbf{u}(t) = \mathbf{e}_\pm \tag{4.6}$$

has a unique solution in $\mathcal{X}_+$ and $\mathcal{X}_-$, respectively.

Remark 3. The asymptotic results (4.6) follow by virtue of the following: for any vector $\mathbf{w}(t)$ satisfying $\mathbf{w}(t) = o(1)$ as $|t| \to \infty$, the assumption (A-III) implies that

$$\mathcal{J}^{-t}\mathbf{w}(t) = o(1), \qquad |t| \to \infty.$$

The solution of the problem (4.6) occupies the rest of this section. To begin with, let us first examine the asymptotic properties of $\mathbf{u}$ separately as $t \to \infty$ and $t \to -\infty$.

Asymptotics as $t \to \infty$. Rewrite Eq. (4.4) as $\mathcal{J}^{-1} \prod_{i=0}^{t-1} (\mathcal{J} + \mathbf{D}(i))\mathbf{g} = \mathbf{e}^{+} + o(1)$, $t \geq 0$. The principal tool for comparing the behavior of the perturbed and unperturbed solutions is the wave operators. Given $\mathbf{g} \in \mathbf{C}^{n}$, we define the *wave operator* $\mathcal{M}_{+}$ by

$$\mathcal{M}_{+}\mathbf{g} = \lim_{t\to\infty} \mathcal{J}^{-t} \prod_{i=0}^{t-1} (\mathcal{J} + \mathbf{D}(i))\mathbf{g} \tag{4.7}$$

whenever this limit exists. Clearly, the range of $\mathcal{M}_{+}$, denoted by $\mathcal{R}(\mathcal{M}_{+})$, is $\mathcal{X}_{+}$. We want to see whether this mapping is injective. Introducing the notation

$$\mathbf{D}_{t} = \mathcal{J}^{-t-1}\mathbf{D}(t)\mathcal{J}^{t}, \qquad t \geq 0, \tag{4.8}$$

we claim that $\mathcal{M}_{+}$, if it exists, can be represented in the form of an infinite product of matrices $\mathbf{I} + \mathbf{D}_{t}$, and we formulate this as a separate lemma:

LEMMA 4.1. *A necessary and sufficient condition for the limit in* (4.7) *to exist for every* $\mathbf{g} \in \mathbf{C}^{n}$ *is the convergence of the infinite matrix product* $\prod_{t=0}^{\infty} (\mathbf{I} + \mathbf{D}_{t})$. *If this is the case then*

$$\mathcal{M}_{+} = \prod_{t=0}^{\infty} (\mathbf{I} + \mathbf{D}_{t}), \tag{4.9}$$

and therefore, $\mathcal{M}_{+}$ is invertible.

Proof. Set $P_{t} = \prod_{k=0}^{t} (\mathbf{I} + \mathbf{D}_{k})$ and $S_{t} = \mathcal{J}^{-t-1} \prod_{k=0}^{t} (\mathcal{J} + \mathbf{D}(k))$. By a straightforward calculation, one finds that $P_{t} = S_{t}$ for all $t \geq 0$ and therefore the assertion holds. ∎

PROPOSITION 4.1. *The function $\mathbf{u} \in \mathcal{H}$ satisfies*

$$\mathbf{u}(t) \sim \mathcal{J}^{t}\mathbf{e}_{+} \tag{4.10}$$

as $t \to \infty$ if and only if the infinite matrix product in (4.9) *converges. If this is true, then the wave operator $\mathcal{M}_{+}$ exists as an isomorphism of $\mathcal{H}$ onto $\mathcal{X}_{+}$. Moreover,*

$$\mathbf{e}_{+} = \mathcal{M}_{+}\mathbf{u}(0). \tag{4.11}$$

is to be a unique solution of (4.6) *as $t \to \infty$.*

Asymptotics as $t \to -\infty$. We now mimic as much as possible the above analysis in order to cover the case $t \to -\infty$. The meaning of the wave operator $\mathcal{M}_-$ is similar. In fact, by virtue of (4.5), for $\mathbf{g} \in \mathbf{C}^n$ we define

$$\mathcal{M}_-\mathbf{g} = \lim_{t \to -\infty} \mathcal{J}^{-t} \left[\prod_{i=t}^{-1} (\mathcal{J} + \mathbf{D}(i)) \right]^{-1} \mathbf{g}, \qquad t < 0 \tag{4.12}$$

provided that the limit exists. It follows that $\mathcal{R}(\mathcal{M}_-) = \mathcal{X}_-$. The foregoing considerations suggest that it would be appropriate to extent the notation for the matrices $\mathbf{D}_t$, given by (4.8), to negative t as well. Then, by similar arguments,

$$\mathcal{M}_-^{-1} = \prod_{t=-\infty}^{-1} (\mathbf{I} + \mathbf{D}_t) \tag{4.13}$$

if and only if this infinite matrix product converges.

PROPOSITION 4.2. *For every* $\mathbf{u} \in \mathcal{H}$, *the asymptotic formula*

$$\lim_{t \to -\infty} \mathcal{J}^{-t}\mathbf{u}(t) = \mathbf{e}_- \tag{4.14}$$

is satisfied if and only if the infinite matrix product in (4.13) *converges. If so, then the wave operator* $\mathcal{M}_-$ *gives the desired isomorphism of* $\mathcal{H}$ *onto* $\mathcal{X}_-$ *and moreover,*

$$\mathbf{u}(0) = \mathcal{M}_-^{-1}\mathbf{e}_-. \tag{4.15}$$

Remark 4. Note that the wave operators $\mathcal{M}_\pm$ are λ-dependent.

Asymptotics of Solutions $\mathbf{u}$ *as* $|t| \to \infty$. We are now in a position to describe the transition between the asymptotically free states $\mathbf{e}_\pm$ as $t \to \pm\infty$. We will see momentarily the utility of Eqs. (4.11) and (4.15).

We employ our preliminary discussion as follows. If we suppose that the two one-sided infinite matrix products $\prod_{t=0}^{\infty} (\mathbf{I} + \mathbf{D}_t)$ and $\prod_{t=-\infty}^{-1} (\mathbf{I} + \mathbf{D}_t)$ converge then the *two-sided* infinite product

$$\prod_{t=-\infty}^{\infty} (\mathbf{I} + \mathbf{D}_t) = \prod_{t=0}^{\infty} (\mathbf{I} + \mathbf{D}_t) \cdot \prod_{t=-\infty}^{-1} (\mathbf{I} + \mathbf{D}_t) \tag{4.16}$$

exists and is nonsingular. We transform this condition into a definition.

DEFINITION 4.1. The matrix $\mathcal{M} = \mathcal{M}_+\mathcal{M}_-^{-1}$ is called the *monodromy matrix* associated with the vector equation (3.8).

It is evident that a monodromy matrix, if it exists, is always unique. The definition implies that

$$\mathcal{M} = \prod_{t=-\infty}^{\infty} (\mathbf{I} + \mathbf{D}_t), \tag{4.17}$$

where the ordering of factors in this infinite matrix product is from the right to the left.

As a conclusion of the above viewpoints, the monodromy matrix (monodromy operator) can indeed be interpreted as follows. In view of (4.11) and (4.15) we obtain

$$\mathbf{e}_+ = \mathcal{M}_+\mathcal{M}_-^{-1}\mathbf{e}_-,$$

and hence $\mathbf{e}_+ = \mathcal{M}\mathbf{e}_-$. Thus, the effect of the perturbation is to change an asymptotically free state which starts out as $\mathbf{e}_-$ near $t = -\infty$ into the asymptotically free state $\mathbf{e}_+$ near $t = +\infty$. The monodromy operator $\mathcal{M}$ defines the transition from $\mathbf{e}_-$ to $\mathbf{e}_+$.

Combining the asymptotic results given in Propositions 4.1 and 4.2 we obtain the following theorem.

THEOREM 4.1. *Consider the vector equation* (3.8) *and assume that the two one-sided infinite matrix products in* (4.16) *converge. Let* $\mathbf{u}(t)$ *be the solution of* (3.8) *with a prescribed initial condition* $\mathbf{u}(0)$. *Then* $\mathbf{u}(t)$ *has the asymptotics* $\mathbf{u}(t) \sim \mathcal{J}^t\mathbf{e}_\pm$ *as* $t \to \pm\infty$. *The matrices* $\mathbf{D}_t$ *are determined by formula* (4.8) *for all* t. *In addition, the vectors* $\mathbf{e}_+$ *and* $\mathbf{e}_-$ *represent the asymptotically free states of the operator* $\mathcal{J} + \mathbf{D}(t)$ as $t \to \pm\infty$ *which are uniquely determined by*

$$\mathbf{e}_\pm = \mathcal{M}_\pm\mathbf{u}(0).$$

The monodromy matrix $\mathcal{M}$ *defined by* (4.17) *is the one-to-one correspondence between* $\mathcal{X}_+$ *and* $\mathcal{X}_-$ *in the sense* $\mathbf{e}_+ = \mathcal{M}\mathbf{e}_-$.

Proof. If we recall the assumptions (A-I)–(A-III), and reapply the change of variable formula (3.6), the identity

$$\mathbf{I} + \mathbf{D}_t = \mathcal{J}^{-t-1}(\mathcal{J} + \mathbf{D}(t))\mathcal{J}^t,$$

clearly assures that $\det(\mathbf{I} + \mathbf{D}_t) \neq 0$ for all t. On the other hand, from the convergence conditions for the infinite products of matrices, both $\mathcal{M}_-$ and $\mathcal{M}_+$ exist and have nonzero determinant. They therefore determine the monodromy matrix $\mathcal{M}$. The rest follows obviously from Propositions 4.1 and 4.2. ∎

The transparency of these results in the case of the system $\mathbf{x}(t+1) = [\mathcal{A} + \mathcal{B}(t)]\mathbf{x}(t)$ now becomes evident. For all $t \in \mathbf{Z}$, we set

$$\mathbf{B}_t = \mathcal{A}^{-t-1}\mathcal{B}(t)\mathcal{A}^t. \tag{4.18}$$

COROLLARY 4.1. *Suppose that the two one-sided infinite matrix products in*

$$\mathcal{N}_-^{-1} = \prod_{t=-\infty}^{-1} (\mathbf{I} + \mathbf{B}_t) \quad \textit{and} \quad \mathcal{N}_+ = \prod_{t=0}^{\infty} (\mathbf{I} + \mathbf{B}_t) \tag{4.19}$$

converge. Then

$$\mathbf{x}(t) \sim \mathcal{A}^t\mathbf{f}_\pm, \qquad t \to \pm\infty, \tag{4.20}$$

where the asymptotally free states $\mathbf{f}_\pm$ *of the perturbed operator* $\mathcal{A} + \mathcal{B}(t)$ *are given by* $\mathbf{f}_\pm = \mathcal{N}_\pm\mathbf{x}(0)$. *The associated monodromy matrix has the form*

$$\mathcal{N} = \mathcal{N}_+\mathcal{N}_-^{-1} = \prod_{t=-\infty}^{\infty} (\mathbf{I} + \mathbf{B}_t). \tag{4.21}$$

As is clear from Eqs. (3.5) and (3.7), $\mathbf{B}_t = \mathbf{V}\mathbf{D}_t\mathbf{V}^{-1}$. Hence, the monodromy matrices $\mathcal{M}$ and $\mathcal{N}$, associated with Eqs. (3.2) and (3.8), respectively, are also similar; i.e., $\mathcal{N} = \mathbf{V}\mathcal{M}\mathbf{V}^{-1}$.

THEOREM 4.2. *Let* $V(t)$ *be a short-range potential. Then both* $\mathcal{N}_\pm$ *exist as respective isomorphic operators of* $\mathcal{H}$ *onto* $\mathcal{X}_\pm$, *respectively. According to Eq.* (4.19), *the monodromy matrix* $\mathcal{N}$, *given by* (4.21), *is the one-to-one correspondence between* $\mathcal{X}_+$ *and* $\mathcal{X}_-$ *described by*

$$\mathbf{f}_+ = \mathcal{N}\mathbf{f}_-. \tag{4.22}$$

Proof. By virtue of Eq. (4.18), it follows that $\det(\mathbf{I} + \mathbf{B}_t) \neq 0$ for all t. The identity $\mathbf{B}_t = \mathbf{V}\mathcal{J}^{-t-1}\mathbf{V}^{-1}\mathcal{B}(t)\mathbf{V}\mathcal{J}^t\mathbf{V}^{-1}$, together with Lemma 2.1, implies that

$$\|\mathbf{B}_t\| \leq \|\mathbf{V}\|\,\|\mathcal{J}^{-1}\|\,\|\mathcal{J}^{-t}\mathbf{D}(t)\mathcal{J}^t\|\,\|\mathbf{V}^{-1}\| \leq \kappa_2(\mathbf{V})\|\mathbf{D}(t)\|,$$

where $\kappa_2(\mathbf{V}) = \|\mathbf{V}\|\,\|\mathbf{V}^{-1}\|$ is the condition number corresponding to the spectral norm of the matrix $\mathbf{V}$. If V is a short-range potential then, according to (3.11),

$$\sum_{t=-\infty}^{\infty} \|\mathbf{B}_t\| < \infty.$$

Hence, the two infinite matrix products in (4.19) converge. The rest of the proof is obvious. ∎

The asymptotics of $\mathbf{x}(t)$ imply the corresponding asymptotics of the eigenfunctions $y \in l^\infty(\mathbf{Z})$ of the operator $\mathbf{L}^{(k)}$.

COROLLARY 4.2. *There are unique t-independent coefficients* $\alpha_i(\lambda), \overline{\alpha}_i(\lambda)$, $i = 1, ..., n$, *such that*

$$y(t, \lambda) \sim \sum_{i=1}^{n} \alpha_i(\lambda) z_i(\lambda)^t, \qquad t \to -\infty, \tag{4.23}$$

$$y(t, \lambda) \sim \sum_{i=1}^{n} \overline{\alpha}_i(\lambda) z_i(\lambda)^t, \qquad t \to \infty, \tag{4.24}$$

where

$$\boldsymbol{\alpha}(\lambda) = [\alpha_1(\lambda), \alpha_2(\lambda), ..., \alpha_n(\lambda)]^\tau = \mathbf{V}_\lambda^{-1}\mathbf{f}_-(\lambda)$$

and

$$\overline{\boldsymbol{\alpha}}(\lambda) = [\overline{\alpha}_1(\lambda), \overline{\alpha}_2(\lambda), ..., \overline{\alpha}_n(\lambda)]^\tau = \mathbf{V}_\lambda^{-1}\mathbf{f}_+(\lambda),$$

provided that $V(t) = \{p_j(t)\}$ *is short-range. Moreover,*

$$\overline{\boldsymbol{\alpha}} = \mathbf{V}^{-1}\mathcal{M}\mathbf{V}\boldsymbol{\alpha}. \tag{4.25}$$

5. SHIFT INFLUENCE ON THE MONODROMY

To solve the scattering problem of a two-sided difference operator $\mathbf{L}$, the previous findings suggest that it may be appropriate to examine the behavior of the monodromy operator under the translation

$$\mathbf{L} \mapsto \mathbf{L}^{(k)} = \mathbf{L}\mathbf{E}^k. \tag{5.1}$$

Let $\mathcal{M}(\lambda)$ and $\mathcal{M}(\lambda)^{(k)}$ be the respective monodromy matrices associated with $\mathbf{L}$ and $\mathbf{L}^{(k)}$. We consider the diagram

$$\begin{tikzcd} \mathbf{L} \arrow[r] \arrow[d, dashed] & \mathbf{L}^{(k)} \arrow[d] \\ \mathcal{M}(\lambda) & \mathcal{M}(\lambda)^{(k)} \arrow[l] \end{tikzcd}$$

If we establish $\mathcal{M}(\lambda) \sim \mathcal{M}(\lambda)^{(k)}$ with respect to a similarity transformation depending on k (the number of steps in the shift), then the above diagram necessarily commutes. For this purpose let us return to a detailed consideration of the eigenvalue problem for the operator $\mathbf{L}$ given by (2.1), i.e., $\mathbf{L}y(t) = \lambda y(t)$, assuming $\tilde{p}_m \equiv 1$. Its corresponding matrix formulation is given by

$$\mathbf{z}(t+1, \lambda) = [\mathcal{P}_\lambda + \mathcal{Q}(t)]\mathbf{z}(t, \lambda), \tag{5.2}$$

where

$$\mathbf{z}(t, \lambda) = \begin{bmatrix} y(t-k) \\ y(t-k+1) \\ \vdots \\ y(t+m-1) \end{bmatrix}$$

and $m + k = n$. The matrices $\mathcal{P}_\lambda$ and $\mathcal{Q}(t)$ (the latter is independent of λ) have the form

$$\mathcal{P}_\lambda = \begin{bmatrix} 0 & 1 & 0 & \cdots & \cdots & \cdots & 0 \\ 0 & 0 & 1 & 0 & \cdots & \cdots & 0 \\ \vdots & & \ddots & \ddots & & & \vdots \\ \vdots & & & & & & \\ \vdots & & & & & \ddots & 0 \\ 0 & \cdots & \cdots & & & \ddots & 1 \\ -p_{-k} & -p_{-k+1} & \cdots & -p_{-1} & \lambda - p_0 & \cdots & -p_{m-1} \end{bmatrix} \tag{5.3}$$

and

$$\mathcal{Q}(t) = \mathbf{e}_n \mathbf{p}^T(t)$$

with $\mathbf{p}(t) = [-p_{-k}(t), -p_{-k+1}(t), \ldots, -p_{m-1}(t)]^T$. We employ the shift (5.1) as

$$(\mathbf{L}y(t) = \lambda y(t)) \overset{\mathbf{E}^k}{\mapsto} (\mathbf{L}^{(k)}y(t) = \lambda y(t+k))$$

or equivalently,

$$\mathbf{z}(t+k+1, \lambda) = [\mathcal{P}_\lambda + \mathcal{Q}(t)]\mathbf{z}(t+k, \lambda). \tag{5.4}$$

Thus, the matrices $\mathcal{P}_\lambda$ and $\mathcal{Q}(t)$ *remain invariant* under the shift applied to the operator $\mathbf{L}$ from the right.

We remark that, in analogy to Proposition 3.1, a necessary condition for all the eigenvalues of $\mathcal{P}_\lambda$ to be of magnitude one is that the constants p_j in (5.3) satisfy

$$|p_j| \leq \binom{n}{j+k}; \qquad -k \leq j \leq m-1 \text{ and } j \neq 0$$

and

$$|\lambda - p_0| \leq \binom{n}{k}.$$

We set $\mathbf{x}(t) = \mathbf{E}^k \mathbf{z}(t)$. Then, the above scattering analysis brings us to the following: The asymptotic formula (4.20) implies that the asymptotics of the solution of the matrix eigenvalue equation (5.4), in terms of $\mathbf{x}(t)$, can be characterized as

$$\mathbf{x}(t) \sim \mathcal{P}^t \mathbf{f}_\pm, \qquad t \to \pm\infty,$$

where $\mathbf{f}_\pm = \mathcal{M}_\mp^{(k)} \mathbf{x}(0)$. This further implies that $\mathbf{z}(t) \sim \mathcal{P}^{t-k}\mathbf{f}_\pm$ as $t \to \pm\infty$. According to (5.1), the asymptotically unperturbed states of the vector equation (5.2) are given by

$$\mathbf{m}_\pm = \mathcal{P}^{-k}\mathbf{f}_\pm. \tag{5.5}$$

On the other hand,

$$\mathbf{m}_+ = \mathcal{M}\mathbf{m}_- . \tag{5.6}$$

The desired similarity between the matrices $\mathcal{M}$ and $\mathcal{M}^{(k)}$ is therefore described by the following assertion.

THEOREM 5.1. *Let* $\mathbf{L}$ *be the nth order two-sided linear difference operator of the eigenvalue problem* (2.2), *and let* $\mathbf{L}^{(k)} = \mathbf{L}\mathbf{E}^k$. *The corresponding monodromy matrices* $\mathcal{M}$ *and* $\mathcal{M}^{(k)}$ *are similar; i.e.,*

$$\mathcal{M} = \mathcal{P}^{-k}\mathcal{M}^{(k)}\mathcal{P}^k, \tag{5.7}$$

where the t-independent matrix $\mathcal{P}$ *is given by* (5.3).

Proof. Follows from (5.5) and (5.6). ∎

6. APPLICATION TO THE TODA LATTICE

The nonlinear Toda lattice represents a system of unit masses connected by nonlinear springs whose restoring force is exponential. It was invented and extensively studied by Toda [13], who discovered a number of remarkable explicit solutions for both the periodic and the infinite lattice. The latter is approximated by the KdV equation in one of the possible long wave continuum limits. Using the discrete inverse scattering theory of Case and Kac [4] and Case [5], Flaschka [9] developed an inverse-scattering method for solving the lattice.

Here, stimulated by Flaschka's work, we incorporate the preceding results in order to recover the existence of the monodromy matrix for the discrete second-order Schrödinger scattering problem that is particularly associated with the lattice. The approach is based on the discrete version of Lax's generalized technique for solving certain nonlinear partial differential equations; see [11]. (The Hilbert space $l^2(\mathbf{Z})$ will be needed.)

The equations of motion for the Toda lattice,

$$\begin{aligned} \dot{Q}_n &= P_n \\ \dot{P}_n &= e^{-(Q_n - Q_{n-1})} - e^{(Q_{n+1} - Q_n)}, \end{aligned} \tag{6.1}$$

where a dot stands for differentiation with respect to time, are derivable from the Hamiltonian

$$\mathbf{H} = \sum_{n=-\infty}^{\infty} \left\{ \frac{1}{2} P_n^2 + (e^{-(Q_n - Q_{n-1})} - 1) \right\},$$

in which Q_n is the displacement of the nth mass from equilibrium and P_n is its momentum. (For the convenience of the reader, in this section we shall denote the space variable by n, $n \in \mathbf{Z}$, whereas the time variable will be t, $t \geq 0$.) To simplify the notation, we omit the dependence on the time variable.

Considering (6.1) for a lattice infinitely long in both directions we set

$$\begin{aligned} a(n) &= \tfrac{1}{2}e^{(Q_{n+1}-Q_n)/2} \\ b(n) &= -P_n/2, \qquad n \in \mathbf{Z}. \end{aligned} \tag{6.2}$$

We emphasize that $a(n)$ and $b(n)$ depend smoothly on a parameter t. In addition, it is evident that $a(n) > 0$ for all n.

We turn our consideration to motion which is confined in some finite region of the lattice, assuming no motion in the distance. Therefore, for $|n| \gg 1$, we have

$$Q_{n+1} - Q_n = 0, \qquad P_n = 0$$

and hence, we can think of $a(n) \to \frac{1}{2}$ and $b(n) \to 0$ rapidly as $|n| \to \infty$.

To derive the connection between the Toda lattice and the Schrödinger discrete second-order eigenvalue problem, we introduce the self-adjoint operator $\mathbf{L}$ and the skew-adjoint operator $\mathbf{B}$, acting on $H = l^2(\mathbf{Z},)$ by the formulas

$$\begin{aligned} \mathbf{L}y(n) &= a(n-1)y(n-1) + b(n)y(n) + a(n)y(n+1) \\ \mathbf{B}y(n) &= a(n)y(n+1) - a(n-1)y(n-1). \end{aligned}$$

In view of the setting given by (6.2), the *Lax representation* of the equations of motion (6.1) is given by

$$\frac{d\mathbf{L}}{dt} = [\mathbf{B}, \mathbf{L}] = \mathbf{BL} - \mathbf{LB}.$$

It implies that all the eigenvalues λ of $\mathbf{L}$ are time-independent (they are constants of the motion). In addition, since the operator $\mathbf{L}$ is self-adjoint, they are also real.

From the asymptotic behavior of the coefficients $a(n)$ and $b(n)$ of the operator $\mathbf{L}$, the eigenvalue equation

$$\mathbf{L}\varphi(n) = \lambda\varphi(n) \tag{6.3}$$

is asymptotically close to

$$\tfrac{1}{2}[\varphi(n-1)+\varphi(n+1)]=\lambda\varphi(n). \tag{6.4}$$

The translation $\mathbf{L}\mapsto\mathbf{LE}$ transforms (6.3) to

$$\varphi(n+2)+\tilde{b}(n)\varphi(n+1)+\tilde{a}(n-1)\varphi(n)=\lambda\frac{1}{a(n)}\varphi(n+1), \tag{6.5}$$

where $\tilde{b}(n)=b(n)/a(n)$ and $\tilde{a}(n-1)=a(n-1)/a(n)$, satisfying $\tilde{b}(n)\to 0$ and $\tilde{a}(n)\to 1$ rapidly as $|n|\to\infty$. Hence, Eq. (6.5) can be written as

$$\varphi(n+2)-2\lambda\varphi(n+1)+\varphi(n)=v_1(n)\varphi(n)+v_2(n)\varphi(n+1) \tag{6.6}$$

for some functions $v_1(n)$ and $v_2(n)$ that decay sufficiently fast as $|n|\to\infty$. In our notation from Section 3, the matrix formulation of Eq. (6.6) gives

$$\mathbf{x}(n+1)=[\mathcal{A}_\lambda+\mathcal{B}(n)]\mathbf{x}(n), \tag{6.7}$$

where

$$\mathbf{x}(n)=\begin{bmatrix}\varphi(n)\\ \varphi(n+1)\end{bmatrix}$$

and

$$\mathcal{A}_\lambda=\begin{bmatrix}0 & 1\\ -1 & 2\lambda\end{bmatrix}\qquad\text{and}\qquad \mathcal{B}(n)=\begin{bmatrix}0 & 0\\ v_1(n) & v_2(n)\end{bmatrix}.$$

Set $\lambda=\frac{1}{2}(z+z^{-1})$. In light of our previous results, the Jost solutions $\varphi_j(n,z)$ of (6.6) are characterized by

$$\begin{aligned}\varphi_1(n,z)&\sim z^n &&\text{as } n\to\infty\\ \varphi_2(n,z)&\sim z^{-n} &&\text{as } n\to-\infty.\end{aligned}$$

It is evident that, for $|z|=1$, the pairs of functions $\{\varphi_1(n,z),\varphi_1(n,z^{-1})\}$ and $\{\varphi_2(n,z),\varphi_2(n,z^{-1})\}$ form a distinguished basis of solutions for the difference operator $\mathbf{L}-\lambda\mathbf{I}$ as $n\to\infty$ and $n\to-\infty$, as long as $z\neq\pm 1$. We have

$$\left.\begin{aligned}\varphi_1(n,z) &\to z^n\\ \varphi_1(n,z^{-1}) &\to z^{-n}\end{aligned}\right\}\text{as } n\to\infty; \qquad \left.\begin{aligned}\varphi_2(n,z) &\to z^{-n}\\ \varphi_2(n,z^{-1}) &\to z^{n}\end{aligned}\right\}\text{as } n\to-\infty.$$

The *scattering matrix* $\mathbf{S}(\lambda)$ is defined as an algebraic transformation that relates the asymptotics of the Jost solutions as $t \to \pm\infty$. We have

$$\begin{bmatrix}\varphi_1(n,z)\\ \varphi_1(n,z^{-1})\end{bmatrix} = \mathbf{S}(\lambda)\begin{bmatrix}\varphi_2(n,z)\\ \varphi_2(n,z^{-1})\end{bmatrix}, \qquad n\to\infty, \tag{6.8}$$

where

$$\mathbf{S}(\lambda) = \begin{bmatrix}\alpha(z) & \beta(z)\\ \beta(z^{-1}) & \alpha(z^{-1})\end{bmatrix}, \qquad \lambda = \frac{1}{2}(z+z^{-1})$$

for some functions $\alpha(z)$ and $\beta(z)$ satisfying the symmetry conditions

$$\alpha(\bar{z}) = \overline{\alpha(z)}, \qquad \beta(\bar{z}) = -\overline{\beta(z)}$$

and

$$\alpha(z)\alpha(z^{-1}) = 1 + \beta(z)\beta(z^{-1}).$$

For $|z| = 1$, we have $|\alpha(z)|^2 = 1 + |\beta(z)|^2$. It follows that $\mathbf{S}(\lambda)$ is *unitary*. Note that, since $a(n)$ and $b(n)$ in (6.2) are time dependent, the functions α and β also include time.

THEOREM 6.1. *The monodromy matrix associated with Eq.* (6.6) *is* similar *to* $\mathbf{S}(\lambda)$.

Proof. The asymptotics of the general solution $y(n, z)$ of Eq. (6.6) for $|z| = 1$ can be characterized as

$$\begin{aligned} y(n,z) &\sim \gamma_1(z)z^{-n} + \gamma_2(z)z^n, & n&\to-\infty\\ y(n,z) &\sim \delta_1(z)z^{n} + \delta_2(z)z^{-n}, & n&\to+\infty.\end{aligned}$$

It follows that

$$\begin{bmatrix} \delta_1(z) \\ \delta_2(z) \end{bmatrix} = \mathbf{S}(\lambda) \begin{bmatrix} \gamma_1(z) \\ \gamma_2(z) \end{bmatrix}. \tag{6.9}$$

On the other hand, by virtue of the asymptotic formulas (4.23) and (4.24), we obtain

$$\begin{bmatrix} \gamma_1(z) \\ \gamma_2(z) \end{bmatrix} = \mathbf{V}_\lambda^{-1}\mathbf{f}_-(\lambda)$$

and

$$\begin{bmatrix} \delta_1(z) \\ \delta_2(z) \end{bmatrix} = \mathbf{V}_\lambda^{-1}\mathbf{f}_+(\lambda)$$

where $\mathbf{f}_\pm(\lambda)$ are the asymptotically free states of the vector equation (6.7), related by $\mathbf{f}_+(\lambda) = \mathcal{M}(\lambda)\mathbf{f}_-(\lambda)$ with respect to the corresponding monodromy matrix $\mathcal{M}(\lambda)$. It is now a routine matter to check that

$$\mathcal{M}(\lambda) = \mathbf{V}_\lambda \mathbf{S}(\lambda)\mathbf{V}_\lambda^{-1} \tag{6.10}$$

or equivalently,

$$\mathbf{S}(\lambda) = \mathbf{V}_\lambda^{-1}\left[\prod_{n=-\infty}^{\infty} (\mathbf{I} + \mathbf{B}_n)\right]\mathbf{V}_\lambda$$

with $\mathbf{B}_n = \mathcal{A}^{-t-1}(\lambda)\mathbf{B}(n)\mathcal{A}^t(\lambda)$, where in particular

$$\mathbf{V}_\lambda = \begin{bmatrix} 1 & 1 \\ z & z^{-1} \end{bmatrix}.$$

(Compare with (4.25)!) ∎

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REFERENCES

1. M. J. ABLOWITZ AND H. SEGUR, "Solitons and the Inverse Scattering Transform," SIAM, Philadelphia, 1981.
2. L. V. AHLFORS, "Complex analysis," McGraw-Hill, New York, 1979.
3. R. BEALS, P. DEIFT, AND C. TOMEI, "Direct and Inverse Scattering on the Line," Amer. Math. Soc., Providence, RI, 1988.
4. K. M. CASE AND M. KAC, Discrete version of inverse scattering problem, *J. Math. Phys.* **14** (1973), 594–603.
5. K. M. CASE, On discrete inverse scattering problems, II, *J. Math. Phys.* **14** (1973), 916–920.
6. P. DEIFT AND E. TRUBOWITZ, Inverse scattering on the line, *Comm. Pure Appl. Math.* **32** (1979), 121–251.
7. J. C. EILBECK AND R. FLESCH, Calculation of families of solitary waves on discrete lattices, *Phys. Lett. A.* **149** (1990), 200–202.
8. J. C. EILBECK, "Numerical Studies of Solitons on Lattices," Lecture Notes in Physics, Vol. 393, pp. 143–150, Springer-Verlag, Berlin/New York, 1991.
9. H. FLASCHKA, On the Toda lattice. II. Inverse scattering solutions, *Progr. Theoret. Phys.* **51** (1974), 703–706.
10. J. A. GOLDSTEIN, "Semigroups of Linear Operators and Applications," Oxford Univ. Press, New York, 1985.
11. P. D. LAX, Integrals of nonlinear equations of evolution and solitary waves, *Comm. Pure Appl. Math.* **21** (1968), 467–490.
12. K. S. MILLER, "Linear Difference Equations," Benjamin, New York, 1968.
13. M. TODA, "Theory of Nonlinear Lattices," Springer Series in Solid State Sciences, Vol. 20, Springer-Verlag, Berlin/New York, 1978.
14. A. TROO, "Scattering for Linear Difference Operators," Ph.D. thesis, University of Southern California, Los Angeles, 1993.
15. V. E. ZAKHAROV, S. V. MANAKOV, S. P. NOVIKOV, AND L. P. PITAYEVSKY, "Theory of Solitons. The Method of the Inverse Scattering Problem," Nauka, Moscow, 1980. [In Russian]